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Estimates Uniform in Time for the Transition Probability of Diffusions with Small Drift and for Stochastically Perturbed Newton Equations

Sergio Albeverio,[1–3,7] **Astrid Hilbert,**[1,2,4] **and Vassili Kolokoltsov**[1,5,6]

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An estimate uniform in time for the transition probability of diffusion processes with small drift is given. This also covers the case of a degenerate diffusion describing a stochastic perturbation of a particle moving according to the Newton's law. Moreover the random wave operator for such a particle is described and the analogue of asymptotic completeness is proven, the latter in the case of a sufficiently small drift.

KEY WORDS: Diffusion processes; transition probability; Newton's law.

1. INTRODUCTION

In this paper we consider the nondegenerate diffusion process x in $\mathbb{R}^d$ defined by the stochastic differential equation

$$dx = \varepsilon K(x)\, dt + dw \tag{1.1}$$

as well as the degenerate diffusion process (x, v) in $\mathbb{R}^d \times \mathbb{R}^d$ defined by the stochastic system

$$\begin{cases} \dot{x} = v \\ dv = \varepsilon K(x)\, dt + dw \end{cases} \tag{1.2}$$

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[1] Mathematisches Institut, Ruhr-Universität, D-44780 Bochum, Germany.
[2] SFB 237.
[3] BiBoS; CERFIM, Locarno; Acc. Arch., USI, Mendrisio.
[4] Mathematisk Institut, Luleå Tekniska Universitetet, S-97187 Luleå.
[5] Moscow Institute of New Technologies; supported in part by the A. v. Humboldt-Stiftung.
[6] Nottingham Trent University, Nottingham NG1 4BU, United Kingdom.
[7] Institute of Applied Mathematics, University of Bonn, Wegelerstr. 6, D 53115 Bonn; address to which correspondence should be addressed.

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where w is the standard d-dimensional Brownian motion, $\varepsilon \geqslant 0$ is a small parameter, and $K(x)$ is a locally Lipschitz-continuous bounded vector function from $L^2(\mathbb{R}^d, \mathbb{R}^d)$, in order to achieve existence and uniqueness of a strong solution and to meet the assumptions of Proposition 1, respectively.

We give an upper bound for the transition probability of these processes which is uniform in time, under the assumption that ε is small enough. As an application we construct the scattering theory for the evolution described by Eq. (1.2) and prove an analogue of asymptotic completeness. We remark that Eq. (1.2) was studied before [e.g., Potter,[12] McKean,[11] Marcus and Weerasinghe,[10] Albeverio *et al.*,[2] and Albeverio and Klar[3, 4]] where in particular some asymptotic estimates on the solutions were found, and by Albeverio *et al.*,[1] where transience was proven for all $d > 2$ [see also references therein]. We shall study the solution processes for Eq. (1.1) resp. Eq. (1.2) in parallel.

We denote by $\mathscr{P}_\varepsilon(t, y, y_0)$ the density of the transition probability. The letter y stands, in the case of Eq. (1.2), for the pair (x, v) and in the case of Eq. (1.1) for x. $E_y^{t,\varepsilon}$ will denote the expectation (mean value) with respect to the measures defined by the solution process y of Eq. (1.1) resp. Eq. (1.2). Our main result is the following

Theorem 1. If $d \geqslant 3$ then for any $r > 1$ there exists $\varepsilon_0 > 0$ such that for all $0 \leqslant \varepsilon \leqslant \varepsilon_0$ we have the estimate, uniformly in time:

$$\mathscr{P}_\varepsilon(t, y, y_0) \leqslant c(\varepsilon)(\mathscr{P}_0(t, y, y_0))^{1/r} \tag{1.3}$$

with some constant $c(\varepsilon)$ and where

$$\mathscr{P}_0(t, y, y_0) = \frac{1}{(2\pi t)^{d/2}} \exp\left\{-\frac{(y - y_0)^2}{2t}\right\} \tag{1.4}$$

for the process in Eq. (1.1) and

$$\mathscr{P}_0(t, x, v, x_0, v_0) = \frac{3^{d/2}}{\pi^d t^{2d}} \exp\left\{-\frac{1}{2t}|v_0 - v|^2 - \frac{6}{t^3}\left|x_0 - x - \frac{t}{2}(v_0 + v)\right|^2\right\} \tag{1.5}$$

for the process in Eq. (1.2).

We shall prove Theorem 1 in the next section using the following technical result obtained by Albeverio *et al.*[1]

Proposition 1. The characteristic function

$$\chi(\lambda) = E_y^{t,0}\left(\exp\left(i\lambda \int_0^\infty K^2(x(t))\,dt\right)\right)$$

of the random variable $\xi_K \equiv \int_0^\infty K^2(x(t))\,dt$ with respect to the measure defined by the process in Eqs. (1.1) or (1.2) with $\varepsilon = 0$ is analytic in some neighborhood of $\lambda = 0$.

Remark 1. This was proved by Albeverio *et al.*[1] for the process in Eq. (1.2), the proof for Eq. (1.1) is similar and even simpler.

In the literature we find the following pathwise estimates on the growth of Brownian motion and its time integral, which are crucial for proving existence and asymptotic completeness of the random wave operator for the system corresponding to a nonlinear drift.

Proposition 2. [Dvoretski and Erdős[5]] If $d \geqslant 2$, then

$$\liminf_{t \to \infty} \frac{|w(t)|}{t^\beta} = +\infty \tag{1.6}$$

almost surely for any $\beta < (1/2) - (1/d)$ and

Proposition 2′. [Kolokoltsov[6, 8]]. If $d \geqslant 3$, then

$$\liminf_{t \to \infty} \frac{\left|\int_0^t w(s)\,ds\right|}{t^\beta} = +\infty \tag{1.7}$$

almost surely for any $\beta < (3/2) - (1/d)$.

One can generalize Propositions 2 and 2′ for processes in Eqs. (1.1) and (1.2) following almost literally the line of arguments used by Kolokoltsov.[6, 8] Inserting the estimate in Eq. (1.3) we are able to replace Eq. (2.3) in Kolokoltsov,[6] which is essential for the Borel Cantelli argument in the proof, by a corresponding inequality. In this way we obtain a proof of the following:

Theorem 2. If $d \geqslant 3$ then for any $\beta < (1/2) - (1/d)$ (respectively $\beta < (3/2) - (1/d)$) there exists $\varepsilon_0 > 0$ such that for any $0 \leqslant \varepsilon \leqslant \varepsilon_0$ the solution x of the process in Eq. (1.1) (resp. Eq. (1.2)) has the property

$$\liminf_{t \to \infty} \frac{|x(t)|}{t^\beta} = +\infty \tag{1.8}$$

In Albeverio *et al.*,[1] we have used Proposition 2 to prove the existence of the wave operator for the process in Eq. (1.2) in the case of a coefficient K ("force") decreasing at infinity. Namely, we have proved the following

Proposition 3. Suppose that for some $\alpha > 12/7$

$$|K(x)| \leqslant c\,|x|^{-\alpha} \qquad \text{for all} \quad x \in \mathbb{R}^d \tag{1.9}$$

$$|K(x_1) - K(x_2)| \leqslant Dr^{-\alpha}\,|x_1 - x_2| \tag{1.10}$$

for all $x_1, x_2 \in \mathbb{R}^d$ with $|x_1|, |x_2| > r > 0$. Then for $\varepsilon > 0$ and for any given $y_\infty = (x_\infty, v_\infty) \in \mathbb{R}^{2d}$, $d \geqslant 3$, there exists a unique solution $\tilde{y} := (\tilde{x}, \tilde{v})$ of Eq. (1.2) such that with probability one the map $y_\infty \mapsto \tilde{y}(0)$ is injective, and $\tilde{y}$ obeys almost surely

$$\lim_{t \to +\infty} (\tilde{v}(t) - w(t) - v_\infty) = 0 \tag{1.11}$$

$$\lim_{t \to +\infty} \left(\tilde{x}(t) - \int_0^t w(s)\, ds - x_\infty - v_\infty t \right) = 0 \tag{1.12}$$

Using Eq. (1.6) instead of Eq. (1.7) we can easily obtain the analogue of Proposition 3 for the process in Eq. (1.1), namely Proposition 4.

Proposition 4. Let $d \geqslant 3$, and assume K satisfies Eq. (1.9), Eq. (1.10) for some $\alpha > ((1/2) - (1/d))^{-1}$. Then for any $\varepsilon > 0$ and any $y_\infty \in \mathbb{R}^d$ there exists a solution $\tilde{y}$ of Eq. (1.1) such that the map $y_\infty \mapsto \tilde{y}(0)$ is injective with probability one, and $\tilde{y}$ obeys almost surely

$$\lim_{t \to +\infty} (\tilde{y} - w(t) - y_\infty) = 0 \tag{1.13}$$

Propositions 3 and 4 state thus that for any "free" solution (i.e., a solution for the case $\varepsilon = 0$) of Eq. (1.1) resp. Eq. (1.2) there exists for any $\varepsilon > 0$ a unique solution of Eq. (1.1) resp. Eq. (1.2) having the same asymptotic behavior at infinity as a given "free" one in the sense that the map $y_\infty \mapsto \tilde{y}(0)$ is well defined and injective. In this paper we prove (for sufficiently small ε) the converse result, namely

Theorem 3. Let the assumptions of Proposition 4 (resp. Prop. 3) for the process in Eq. (1.1) (resp. Eq. (1.2)) be satisfied. Then there exists an $\varepsilon_0 > 0$ such that for any $0 \leqslant \varepsilon \leqslant \varepsilon_0$ and each solution $\tilde{y}$ of Eq. (1.1) resp. Eq. (1.2) with given initial condition $\tilde{y}(0) = y_0$ there exist (random) y_∞ ($y_\infty \in \mathbb{R}^d$ for Eq. (1.1) and $y_\infty = (x_\infty, v_\infty) \in \mathbb{R}^{2d}$ for Eq. (1.2)) such that Eq. (1.13) respectively Eq. (1.11), Eq. (1.12) hold with probability one.

Remark 2. In terms of scattering theory one can say that Propositions 3 and 4, state the existence of the random wave operator Ω_+ defined by $\Omega_+: y_\infty \mapsto \tilde{y}(0)$. Theorem 3 corresponds to the asymptotic completeness of these operators in the sense that Ω_+ is almost everywhere surjective under the assumptions of Theorem 3, i.e., $\Omega_+(\omega): y_\infty(\omega) \mapsto \tilde{y}(0)(\omega)$ is surjective for almost all paths ω.

2. PROOF OF THEOREM 1

The basic idea of this proof is essentially contained in our paper Albeverio *et al.*[1] Namely, in order to obtain the estimate for the transition probability $\mathscr{P}_\varepsilon(t, y, A)$ for some disk A we use the Girsanov density of the distribution of the nonlinear process y defined by the Eq. (1.1) resp. Eq. (1.2) for $\varepsilon \neq 0$ with respect to the known distribution in Eq. (1.4) resp. Eq. (1.5) of the corresponding linear i.e., "free" process (given by Eq. (1.1) resp. Eq. (1.2) with $\varepsilon = 0$). More precisely we have

$$\mathscr{P}_\varepsilon(t, y, A) = E_y^{t,0}[\chi_A(y_t)\, \varphi_t(y)] \tag{2.1}$$

where on the right-hand side the function χ_A is the indicator of the set A, the expectation is taken with respect to the probability distribution of the "free" process (with $\varepsilon = 0$) and the density $\varphi_t(y)$ is given by the formula

$$\varphi_t(y) = \exp\left\{\varepsilon \int_0^t K(x_s)\, dw_s - \frac{\varepsilon^2}{2} \int_0^t |K|^2 (x_s)\, ds\right\}$$

(we recall that $y = (x, v)$ for the process in Eq. (1.2) resp. $y = x$ for the process in Eq. (1.1) here taken for $\varepsilon = 0$). The fundamental Eq. (2.1) is the standard Girsanov theorem [see, e.g., Lipser and Shiryaev,[9] for the case of the process in Eq. (1.1) and for Eq. (1.2) is a corollary [obtained in Albeverio *et al.*[2]] of a generalized Girsanov theorem.

To estimate Eq. (2.1) we use Hölder's inequality with $r, q \geqslant 1$, $(1/r) + (1/q) = 1$:

$$E_y^{t,0}[\chi_A(y_t)\, \varphi_t(y)] \leqslant \|\chi_A(y_t)\|_r\, \|\varphi_t(y)\|_q \tag{2.2}$$

where $\|\ \|_s$ is the s-norm with respect to $E_y^{t,0}$.

Set $\varphi_t^q(y) = \exp(F_1 - F_2)$ with $F_1 = q \int_0^t \varepsilon K(x_s)\, dw_s$, and $F_2 = (q/2) \times \int_0^t \varepsilon^2 |K(x_s)|^2\, ds$. By Schwarz's inequality, we have now

$$\begin{aligned}\|\varphi_t^q\|_1 &= \|\exp(F_1 - 2qF_2)\exp((2q-1)\, F_2)\|_1 \\ &\leqslant \|\exp(F_1 - 2qF_2)\|_2 \cdot \|\exp(2q-1)\, F_2\|_2\end{aligned}$$

The first term in the last expression is equal to one, because

$$\exp(2(F_1 - 2qF_2)) = \exp\left\{\int_0^t 2q\varepsilon K(x_s)\, dw_s - \tfrac{1}{2}\int_0^t |2q\varepsilon K|^2 (x_s)\, ds\right\}$$

is a martingale and the second term

$$E_y^{t,0}\left(\exp\left\{(2q-1)\, q\varepsilon^2 \int_0^t K^2(x_s)\, ds\right\}\right)$$

is bounded uniformly in t for $\varepsilon \leqslant \varepsilon_0$ with some ε_0 (depending on q), due to Proposition 1. Therefore, we obtain from Eqs. (2.1) and (2.2) that

$$\mathscr{P}_\varepsilon(t, y, A) \leqslant c(\varepsilon) \sqrt[r]{\mathscr{P}_0(t, y, A)}$$

for any $r > 1$ and $0 \leqslant \varepsilon \leqslant \varepsilon_0(r)$. This inequality being true for all disks A obviously implies Eq. (1.3) for densities. □

3. PROOF OF THEOREM 3

We shall limit ourselves to indicate the proof for the process in Eq. (1.2) (for Eq. (1.1) the proof is in fact even simpler). Let us rewrite the system of Eq. (1.2) in terms of the variables $X(t) \equiv x(t) - \int_0^t w(s)\, ds$ and $P(t) \equiv v(t) - w(t)$, obtaining

$$\begin{cases} \dot{X}(t) = P(t) \\ \dot{P}(t) = \varepsilon K\left(X(t) + \int_0^t w(s)\, ds\right) \end{cases}$$

For any solution of this system, due to Eq. (1.8), there exists (almost surely) a time T such that $|x(t)| = |X(t) + \int_0^t w(s)\, ds| \geqslant ct^\beta$ for $t > T$, $\beta < (3/2) - (1/d)$ and some constant c. Then (due to Eq. (1.9))

$$\left|K\left(X(t) + \int_0^t w(s)\, ds\right)\right| < \frac{c}{t^{\alpha\beta}}$$

for $t > T$ and therefore

$$P(t) - P(0) = \varepsilon \int_0^t K\left(X(\tau) + \int_0^\tau w(s)\, ds\right) d\tau$$

has a limit P_∞ depending on the path, as $t \to +\infty$, if $\alpha\beta > 1$. Furthermore, if $\alpha\beta > 2$ we have

$$P(t) - P(0) - P_\infty = -\int_t^\infty \varepsilon K\left(X(\tau) + \int_0^\tau w(s)\,ds\right) d\tau \tag{3.1}$$

and therefore

$$|P(t) - P(0) - P_\infty| < \frac{c}{t^{\alpha\beta - 1}}$$

for sufficiently large t. Moreover (by adding and subtracting terms) we have

$$\begin{aligned} X(t) &= X(0) + \int_0^t P(\tau)\,d\tau \\ &= X(0) + t(P_\infty + P(0)) + \int_0^t (P(\tau) - P(0) - P_\infty)\,d\tau \end{aligned}$$

and we find by Eq. (3.1) that $X(t)$ and $P(t)$ decay in a way that $x(t)$, $v(t)$ have the asymptotics required in the theorem. □

As a concluding remark, let us mention that the systems of the form in Eq. (1.2) are presently intensively studied in connection with their applications in the theory of partial differential stochastic equations, see for instance Truman and Zhao[13] or Kolokoltsov.[7]

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